# Inviscid Flow through Cascades in Oscillatory and Distorted Flow

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**The present paper deals with the flow through a staggered cascade of airfoils in which unsteady disturbances from the upstream are swept down with the flow, as in the case of unsteady or distorted inlet flow conditions in an axial flow compressor. In the oscillatory case, unsteady normal velocity fluctuations which are carried down with the steady-state flow, like traveling waves, cause varying angles of attack along the chord lengths of the airfoils. Disturbances due to steady-state circumferential inlet distortion can be decomposed into Fourier components of oscillatory flows with some phage lag between the blades. The problem has been formulated under the assumptions of incompressible potential flow. An approximate method of solution has been developed for the integral equations involved and numerical results have been obtained for oscillatory flows. The unsteady lift coefficient of the airfoils has been obtained as a function of the frequency of the oscillations. The amplitude of the fluctuating lift for a cascade decreases from the steady state value with increasing frequency more slowly than it does for a single airfoil. The quantitative behavior of this has been studied for different values of stagger angle and solidity of the cascade.**

## Nomenclature‡

$a$ = distance between the trailing edges of two adjacent airfoils of a cascade
$c$ = semichord length of an airfoil
$C_L$ = lift coefficient defined in Eq. (28)
$H$ = transfer function defined in Eq. (29)
$i$ = $(-1)^{1/2}$
$\mathbf{i}_x$ = unit vector in $x$ direction
$\mathbf{i}_y$ = unit vector in $y$ direction
$I$ = $I(x)$ defined in Eq. (19)
$j$ = $(-1)^{1/2}$
$k$ = number of cycles of distortion around the circumference in the distorted flow
$K$ = $K(z)$ defined in Eq. (A13)
$L$ = $L(x)$ defined in Eq. (18)
$N$ = number of blades of the rotor
$\Delta p^*$ = pressure difference—Eq. (26)
$R_1$ = inner radius of the compressor
$R_2$ = outer radius of the compressor
$R_{avg}$ = average radius of the compressor
$t^*$ = time
$t_0^*$ = time for one revolution of the rotor
$T$ = cascade spacing parameter—$a/c$
$v_s^*$ = see Eq. (A1)
$v_p^*$ = see Eq. (A2)
$\mathbf{v}_{ax}$ = axial velocity of flow in the annulus
$\mathbf{v}_{rot}$ = rotational velocity of the airfoils
$\mathbf{v}_{rel}$ = flow velocity relative to the blades
$v_x^*$ = $x^*$component of the velocity field
$v_y^*$ = $y^*$component of the velocity field
$V^*$ = $V^*(z^*)$—Eq. (9)
$V_x^*$ = real part of $V^*$
$V_y^*$ = real part of $iV^*$
$w$ = unsteady normal velocity distribution of the flow relative to the blades
$\bar{w}$ = amplitude of sinusoidal $w$
$W$ = steady-state velocity relative to the blades
$x^*$ = coordinate along the airfoil chord
$y^*$ = coordinate perpendicular to the airfoil chord
$z^*$ = $x^* + iy^*$
$\alpha$ = angle of attack of steady-state flow
$\gamma$ = refer Eq. (16)
$\Gamma$ = refer Eq. (A25)
$\xi^*$ = fixed coordinate in the axial direction of the compressor
$\eta^*$ = coordinate normal to $\xi^*$
$\eta_0^*$ = average circumference of the blade row
$\lambda$ = stagger angle of the cascade
$\mu$ = phase lag between blades-Eq. (24)
$\nu$ = actual frequency
$\omega$ = reduced frequency—$2\pi c\nu/W$
$\omega_0$ = refer Eq. (22)
$\Omega$ = $\Omega(z)$—Ref. Eq. (14)

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‡ Some of the quantities defined above with * superscript have their nondimensional counterparts. Such nondimensional quantities are denoted by the same symbols but without the * superscript.

## I. Introduction

FOR investigating the unsteady and distorted flow effects on axial flow compressors, it is customary to use the quasi-steady-state aerodynamics for describing the stage characteristics.[1] However, this approach by nature is restricted to low-frequency oscillations. In order to estimate the range of validity of such an approach and to account for the dynamic effects on stage characteristics at high frequencies it is important to have a detailed analysis of the unsteady flow through a single row of blades of a compressor.

With the assumption of two-dimensional flow, the flow through a single row is mathematically equivalent to the flow through a staggered cascade of infinitely many airfoils. The present work deals with oscillatory and distorted flows through a staggered cascade of airfoils. There has been some work since around 1950 concerning the unsteady effects on inviscid flow through cascades.

Kemp and Sears[2,3] have studied the aerodynamic interference between a stator and a rotor under the simplifications of two-dimensional incompressible single airfoil theory. They considered the unsteady effects due to the relative motion of the steady-state parts of the flowfields around the stator and the rotor, but neglected the mutual interaction of the circulations around the airfoils of the cascades. Recently Mani[4] generalized the Kemp and Sears problem allowing for the cascade effects and compressibility effects. There have been several other investigations which dealt with unsteady flows through isolated cascades of airfoils and reviews of many of these investigations can be found in Refs. 5–8. Probably

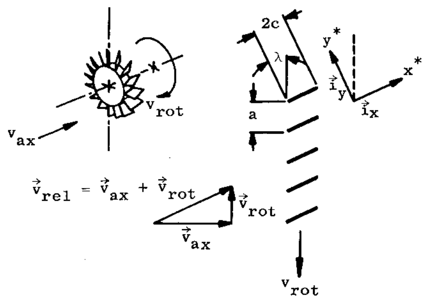


**Fig. 1 Schematic representation of flow velocity relative to the blades; arrowed letters are boldface in text.**

the most general formulation and analysis of the problem is by Meister,[7,8] but his results are far from a suitable form for obtaining practical numerical solutions. Meister[7] considered the problem of unsteady flow through a staggered cascade with a very general type of harmonic oscillations for the blades under incompressible flow assumptions and in Ref. 8 the compressible flow aspects of the problem have been studied.

Unsteady flow due to a cascade of airfoils oscillating about their pivot points in an otherwise uniform flow, which has been studied by many of the past investigators, may be important for flutter analysis. But it is not the same as the flow through a cascade of rigid airfoils in which unsteady disturbances in the upstream are swept down with the flow as in the case of unsteady or distorted inlet flow conditions. We are concerned with the latter problem in the present paper.

Since we are interested in the effects of small unsteady disturbances, we assume that there is a basic steady-state flow through an unloaded staggered cascade of airfoils, over which unsteady normal velocity fluctuations are superimposed. The source of fluctuations is in the upstream and the oscillations in normal velocities relative to the blades are carried down with the main flow like traveling waves and they cause varying angles of attack along the chord lengths of the airfoils. This we call oscillatory flow and is analogous to Küssner's problem[9] of gust flow past an airfoil in two-dimensional unsteady wing theory. Inlet flow oscillations of the type described here are mathematically equivalent to prescribing normal relative velocities of the following form on all airfoils of the cascade, $w(x,t) = \bar{w}e^{j\omega(t-x)}$ where $\bar{w}$ is an arbitrary small quantity, $t$ nondimensional time, $x$ nondimensional space coordinate along the chord length of an airfoil, and $\omega$ the reduced frequency. Steady-state circumferential distortion is also considered which can be decomposed by Fourier series approach into oscillatory flows with some phase lag between relative normal velocity distributions on different airfoils. The relative normal velocity on the $n$th airfoil is of the form $w_n(x,t) = \bar{w}e^{j\omega(t-x) - jn\mu}$ where $\mu$ is a phase lag between two adjacent blades.

Under the assumptions of incompressible inviscid flow, the problem of oscillatory flow is reduced to the solution of a singular integral equation. We present an approximate solution of the integral equation analogous to the solution of singular integral equations in the theory of oscillatory isolated airfoils. This can be used to compute quantities of interest like pressure difference across the cascade. In the present case the pressure difference per unit area across the cascade is equal to the lift per unit area taken on any one airfoil. The unsteady lift coefficient has been obtained as a function of the frequency of the oscillations for different combinations of the parameters of the problem-stagger angle and solidity of the cascade, and numerical results are discussed. This provides a normalized transfer function relating sinusoidal normal velocity disturbances as input and the corresponding lift as output, which could be used for analyzing more general types of incoming flows. The present formulation and method of solution could also be used for studying the unsteady flow caused by oscillating airfoils of a cascade.

## II. Description of Oscillatory and Distorted Flow

Consider an annular channel with inner and outer radii such that $(R_2 - R_1)/R_2 \ll 1$. Within the annulus we have a rotating row of $N$ thin blades with stagger angle $\lambda$ and spacing $T = a/c$, where $a$ is the distance between the leading edges of two adjacent blades and $c$ is the semichord length of a blade. The flow of an incompressible fluid through a thin annulus of the channel can be considered two-dimensional, i.e., the section of the blades is unwrapped into a staggered cascade of infinitely many airfoils in a plane. Let $\mathbf{v}_{rot}$ be the rotating velocity of the airfoils and $\mathbf{v}_{ax}$ be the axial velocity of the fluid in the channel and $\boldsymbol{\varepsilon}$ be the unsteady small disturbance in the axial velocity. With the coordinates fixed on a blade, the relative velocity $\mathbf{v}_{rel}$ of the fluid with respect to the blades is given by (refer to Fig. 1), $\mathbf{v}_{rel} = \mathbf{v}_{rot} + \mathbf{v}_{ax} + \boldsymbol{\varepsilon} = [(v_{ax} + \epsilon)\cos\lambda + v_{rot}\sin\lambda]\mathbf{i}_x + [v_{rot}\cos\lambda - v_{ax}\sin\lambda - \epsilon\sin\lambda]\mathbf{i}_y$ where $\mathbf{i}_x$ and $\mathbf{i}_y$ are the unit vectors along and perpendicular to the blades. In the case of unloaded blades, there is no mean relative angle of attack and

$$v_{rot}\cos\lambda = v_{ax}\sin\lambda \tag{1}$$

Therefore we can write $\mathbf{v}_{rel} = (W + \epsilon\cos\lambda)\mathbf{i}_x - \epsilon\sin\lambda\mathbf{i}_y$ where

$$W = v_{ax}\cos\lambda + v_{rot}\sin\lambda \tag{2}$$

Neglecting the effects of $\epsilon\cos\lambda$ in comparison with $W$, and writing $\epsilon\sin\lambda$ as $w$, we have

$$\mathbf{v}_{rel} = W\mathbf{i}_x - w\mathbf{i}_y \tag{3}$$

Thus, $w$ represents the unsteady normal velocity distribution of the incoming flow relative to the blades.

In the two-dimensional plane of the cascade, let $\xi^*,\eta^*$ be a fixed coordinate system with $\xi^*$ in the axial direction of the compressor and $\eta^*$ normal to it (see Fig. 2). $x^*,y^*$ is a moving coordinate system fixed on a blade, which we call the zeroeth blade, at the center of its chord length, with $x^*$ in the direction of the chord length and $y^*$ perpendicular to it. We can assume that the origins of the two coordinate systems coincide at time $t^* = 0$. The cascade is moving downward in the negative $\eta^*$ direction with velocity $v_{rot}$. At any time $t^*$, the transformation from $\xi^*,\eta^*$ to $x^*,y^*$ is

$$\begin{aligned} \xi^* &= x^*\cos\lambda - y^*\sin\lambda \\ \eta^* &= x^*\sin\lambda + y^*\cos\lambda - v_{rot}t^* \end{aligned} \tag{4}$$

### Oscillatory Flow

In this case we consider the unsteadiness to be imbedded in the incoming flow and it is carried down by the steady-state

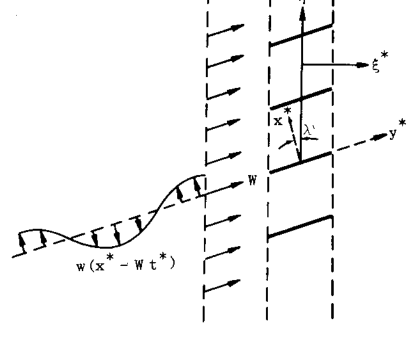


**Fig. 2 Cascade encountering an oscillatory flow of a traveling wave type.**

flow like a traveling wave. The disturbances from upstream travel down, past the cascade, with a velocity $W$ relative to the blades. The wave front is parallel to the plane of the leading edges of the blades.

Thus $w$ is a function of $(\xi^* - v_{ax}t^*)$. Transforming $\xi^*$ into $x^*$ and $y^*$, we can write $w$ as a function of $(x^* - y^* \tan\lambda - Wt^*) \cos\lambda$ where we have used Eqs. (1) and (2). We will consider $w$ to be sinusoidal function of some frequency so that more general functions can be treated by Fourier summation of several sinusoidal components. Thus we choose

$$\begin{aligned} w &= \bar{w}e^{-j2\pi\nu(x^* - y^* \tan\lambda - Wt^*)/W} \\ &= \bar{w}e^{-j\omega(x - y \tan\lambda - t)} \end{aligned} \tag{5}$$

where $\bar{w}$ is a constant, $\nu$ is the actual frequency of the wave and $\omega$ is the reduced frequency defined by

$$\omega = (2\pi c/W)\nu \tag{6}$$

and $t,x$, and $y$ are nondimensional time and space coordinates defined by

$$t = (W/c)t^*, \quad x = x^*/c, \quad y = y^*/c \tag{7}$$

and $j = (-1)^{1/2}$.

The boundary conditions for the present problem are that the velocity as $x^* \to -\infty$ is given by $\mathbf{v}_{rel}$ of Eqs. (3) and (5) and the velocity normal to the blades at the blades is zero. Suppose $v_x^*$ and $v_y^*$ are the components of the velocity field in the directions of $x^*$ and $y^*$, respectively, we define $V_x^*$ and $V_y^*$ as

$$V_x^* = v_x^*, \quad V_y^* = v_y^* + w \tag{8}$$

With this transformation, the problem is reduced to finding a complex velocity field $V^*$ which is written as

$$V^*(z^*,t^*) = V_x^*(x^*,y^*,t^*) - iV_y^*(x^*,y^*,t^*) \tag{9}$$

where

$$z^* = x^* + iy^*, \quad i = (-1)^{1/2}$$

which satisfies the boundary conditions

$$V_x^* \to W, \quad V_y^* \to 0 \text{ as } x^* \to -\infty \tag{10}$$

and at the airfoils,

$$\begin{aligned} V_y^*(x^* + am \sin\lambda, am \cos\lambda, t^*) &= \bar{w} \exp\left[-\frac{j2\pi v}{W}(x^* - Wt^*)\right] \\ &= \bar{w}\, e^{-j\omega(x-t)} \end{aligned} \tag{11}$$

for

$$|x^*| \leq c$$

$$m = 0, \pm 1, \pm 2, \ldots$$

While writing the boundary conditions at the airfoils, we assume that there is no camber or thickness to the airfoils. In general, for small camber and thickness, we have to write this equation separately for upper and lower surfaces of the airfoils and add terms corresponding to camber and thickness.

The aforementioned problem is a particular case of the more general problem formulated in Appendix A. It corresponds to the case of $\mu = 0$ (which implies that $N = 1$), $\alpha = 0$ (symbols are explained in Appendix A), and

$$v_s^*(x^*) = v_p^*(x^*) = \bar{w}e^{-j\omega x} \tag{12}$$

Using the results of Appendix A, we obtain

$$V^* = W\{1 + \bar{w}/W[V(z) + \Gamma\Omega(z)]e^{j\omega t}\} \tag{13}$$

where

$$z = z^*/c$$

$$\Omega(z) = -j\omega \int_1^\infty K(z - \xi)e^{-j\omega(\xi - 1)}\, d\xi \tag{14}$$

$$V(z) = \int_{-1}^1 K(z - \xi)\gamma(\xi)d\xi \tag{15}$$

and $K(z)$ is defined by Eq. (A13), and $\gamma(\xi)$ and $\Gamma$ are solutions of the following equations of which Eq. (16) is a singular integral equation

$$\frac{1}{2\pi} \oint_{-1}^1 \frac{\gamma(\xi)}{\xi - x} d\xi - \int_{-1}^1 L(x - \xi)\gamma(\xi)d\xi + j\omega\Gamma \int_1^\infty I(x - \xi)e^{-j\omega(\xi-1)}\, d\xi = e^{-j\omega x} \tag{16}$$

$$\int_{-1}^1 \gamma(\xi)d\xi = \Gamma \tag{17}$$

where $L(x)$ and $I(x)$ are defined as

$$L(x) = I(x) - 1/2\pi x \tag{18}$$

$$I(x) = Im\left\{\frac{ie^{i\lambda}}{2T}\left[\coth\left(\frac{\pi e^{i\lambda}}{T}x\right) + 1\right]\right\} \tag{19}$$

An approximate solution for $\gamma(\xi)$ and $\Gamma$ has been obtained in Appendix B.

### Distorted Flow

Assume that there is a small steady-state circumferential distortion in the basic steady incoming flowfield. This distortion, as the blades of a rotor rotate through it, will cause unsteady normal velocity distributions relative to the blades which vary along the chord lengths of the airfoils. This will cause varying angles of attack along the chord lengths of the airfoils. Since the circumferential distortion is steady, each of the blades is subjected to the same periodic normal velocity distribution, but with a phase shift.

Let $R_{avg}$ be the average radius of the inner and outer walls of the compressor, $\eta_0^*$ the average circumference of the blade row and $t_0^*$ the time for one revolution of the rotor. Then

$$t_0^* = \eta_0^*/v_{rot}, \; \eta_0^* = 2\pi R_{avg} \tag{20}$$

The velocity of the fluid relative to the blades far upstream is $\mathbf{v}_{rel}$ as given by Eq. (3). This time the disturbance $w$ is due to a variation in the axial velocity far upstream around the circumference. Hence, $w$ is a periodic function in $(\eta^*/\eta_0^*)$. Each of the blades will be subjected to the same periodic normal velocity distribution, but with a phase shift of $t_0^*/N$ between adjacent blades, where $N$ is the number of blades in the blade row. We will consider $w$ to be a sinusoidal function of $k$ cycles around the circumference, where $k$ is an arbitrary positive integer. More general periodic functions can be considered by Fourier summation of sinusoidal components. Thus $w = \bar{w}\; e^{-j2\pi k(\eta^*/\eta_0^*)}$ where $\bar{w}$ is a small constant. Transforming $\eta^*$ into $x^*,y^*$ coordinates, which are fixed on the zeroth blade, by Eq. (4), we get

$$w = \bar{w}\; e^{-jk\omega_0(x + y \cot\lambda - t)} \tag{21}$$

where

$$\omega_0 = \frac{c \sin\lambda}{R_{avg}} = \frac{2\pi c \sin\lambda}{\eta_0^*} = \frac{2\pi \sin\lambda}{TN} \tag{22}$$

and $x,y,t$ are defined by Eq. (7).

Just as in the case of oscillatory flow, the boundary conditions of the problem are that the velocity as $x^* \to -\infty$ is $\mathbf{v}_{rel}$ given by Eqs. (3) and (21) and velocity normal to the blades at the blades is zero. Defining $V^*(z^*,t^*)$ by Eqs. (8) and (9), the problem can be reduced to the boundary conditions $V_x^* \to W$, $V_y^* \to 0$ as $x^* \to -\infty$ and at the airfoils

$$V_y^*(x^* + am \sin\lambda, am \cos\lambda, t^*) = \bar{w}\; e^{-jk\omega_0(x-t) - j\mu m} \tag{23}$$

for,

$$|x^*| \leq c$$

$$m = 0, \pm 1, \pm 2, \ldots\ldots$$

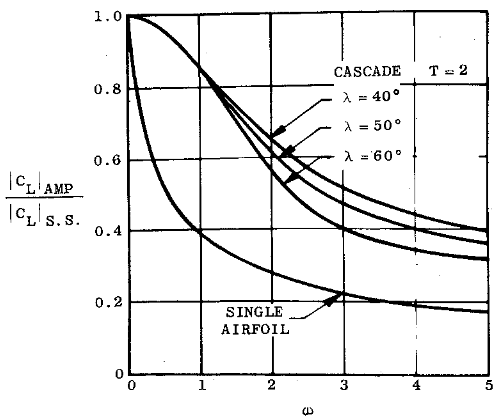


**Fig. 3 Amplitude of normalized lift coefficient; oscillatory flow.**

where

$$\mu = 2\pi ka/\eta_0^* = k\omega_0 T/\sin\lambda = 2\pi k/N,\ T = a/c \quad (24)$$

The $\mu$ term in the aforementioned expressions represents the phase lag of the disturbance between successive blades. Since $k$ is an integer, it can also be seen that the boundary conditions (23) repeat after $N$ blades. Hence we have a periodicity property for the velocity field given by

$$V^*(z^*,t^*) = V^*(z^* + iaNe^{-i\lambda},t^*),\ \text{for all } z^* \quad (25)$$

Now the problem is reduced to a particular case of the general problem formulated in Appendix A, but is more general than that of oscillatory case. It corresponds to the case of $\alpha = 0$ and

$$v_s^*(x^*) = v_p^*(x^*) = \bar{w}\, e^{-jk\omega_0 x}$$

The formal solution to the velocity field is given by Eq. (A24) of Appendix A and the corresponding integral equations for vortex density function are given by Eqs. (A25) and (A32). Solutions to these integral equations can be obtained by a straight forward extension of the method outlined in Appendix B.

Some results obtained for oscillatory case are discussed in the next Section and the distortion case is currently in computational stage.

## III. Discussion of Results for Oscillatory Case

Besides the velocity field, the pressure difference across the cascade is of particular interest. In the present case where the unsteady velocity field is superimposed over the steady-state velocity field with mean angle of attack $\alpha = 0$, it turns out that the pressure difference per unit area across the cascade is equal to the lift per unit area taken at any one airfoil.

For sinusoidal velocity disturbances, the local pressure difference between the pressure and suction sides of an airfoil is of the form

$$\Delta p^*(x;\omega)e^{j\omega t} \quad (26)$$

It can be shown that

$$\Delta p(x;\omega) = \frac{\Delta p^*(x;\omega)}{\rho W \bar{w}} = -\left[\gamma(x;\omega) + j\omega \int_1^x \gamma(\xi;\omega)d\xi\right] \quad (27)$$

where $\gamma(x;\omega)$ which corresponds to the unsteady vortex density along the airfoil, is the solution of Eq. (A32). Integration of Eq. (27) along an airfoil yields the unsteady lift coefficient

$$C_L(t;\omega) = (\bar{w}/W)\, e^{j\omega t} \int_{-1}^{1} \Delta p(x;\omega)dx \quad (28)$$

We may define a normalized transfer function

$$H(\omega) = |C_L|_{\text{amp}}/|C_L|_{\text{s.s.}} = \int_{-1}^{1} \Delta p(x;\omega)dx \Big/ \int_{-1}^{1} \Delta p(x;o)dx \quad (29)$$

where $|C_L|_{\text{amp}}$ denotes the amplitude of $C_L$ and $|C_L|_{\text{s.s.}}$ denotes the steady-state lift coefficient corresponding to the amplitude of the input oscillations. The transfer function $H$ provides the input-output relation between a sinusoidal normal velocity disturbance and the corresponding lift. It can also be looked at as the amplitude of a normalized lift coefficient.

For oscillatory case discussed in the previous section, the solution of Eqs. (16) and (17) as obtained in Appendix B has been computed for different combinations of the parameters involved. A tenth degree polynomial approximation was used for $L(x)$. The amplitude of the normalized lift coefficient as defined in Eq. (29) is plotted against the reduced frequency $\omega$ in Fig. 3 for $T = 2$ (solidity $2/T = 1$), and three different stagger angles. The curve corresponding to a single airfoil ($T = \infty$) is also plotted for comparison. This curve has been obtained using the Küssner's function[9] for oscillatory flow past an isolated airfoil. One noticeable difference between a cascade and a single airfoil is that the amplitude of the unsteady lift coefficient for cascade drops from the steady-state value with increasing $\omega$ much more slowly than it does for a single airfoil. This tends to indicate that the trailing vortices behind the airfoils of a cascade have a mutual cancellation effect. As it can be expected this cancellation effect decreases with increasing stagger angle. It can also be seen from Fig. 4, where similar curves are plotted for a fixed stagger angle 50°, but with different cascade spacings. As the cascade spacing increases the mutual cancellation of the effects of trailing vortices decreases.

The oscillatory case which we have computed considers that the disturbances in the upstream are swept down like traveling waves past stationary airfoils of the cascade. The angles of attack on the airfoils vary from point to point along their chord length as the wave-like disturbance is propagated. For high frequencies of the disturbances there may be several wave lengths per chord length of the airfoil and the fast changing angles of attack cannot build up enough steady-state lift and thus the amplitude of the lift coefficient goes down at high frequencies.

On the other hand, if we have a cascade of airfoils fluttering in a uniform steady-state flow, the situation is different. The unsteady lift coefficient will not only have contribution from the circulatory forces generated, but it will also include the inertial forces of the oscillating airfoils. The unsteady inertial forces dominate the lift at high frequencies and hence the amplitude of lift coefficient goes up with an order of $\omega^2$ for large $\omega$. The mathematical analysis outlined in Appen-

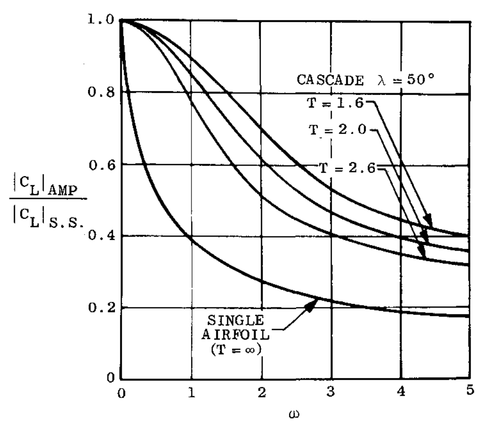


**Fig. 4 Amplitude of normalized lift coefficient; oscillatory flow.**

dices A and B is suitable for studying the case of a cascade of fluttering airfoils with some simple modifications. As an example we computed this case for a cascade of stagger angle 50° and spacing 2 (or solidity 1), the airfoils oscillating about their rear quarter points. The amplitude of normalized lift coefficient is compared with that of a single airfoil in Fig. 5.

In conclusion, we have developed an analytical technique for obtaining approximate solutions of different types of unsteady flow problems through cascades under the assumptions of incompressible, inviscid flow. Results reported in this paper for oscillatory case have been obtained with a reasonable amount of computational labor and they demonstrate that we can study more general cases of circumferentially distorted flows without much more difficulty and we plan to pursue this work further.

## Appendix A: Formulation and Mathematical Reduction of the Unsteady Cascade Flow Problem

In this Appendix we shall formulate the equations for a general unsteady flow superimposed over a basic steady state flow through a cascade with an angle of attack. The unsteadiness of the flowfield is caused by harmonically time dependent normal velocities at the blades which vary along the airfoils. Arbitrary phase difference between adjacent blades is also allowed. The following are the assumptions used: The flow is incompressible and inviscid, the airfoils are thin and the flow disturbances are small so that the process of linearization is valid. The camber and thickness of the airfoils are assumed small, so that the boundary conditions can be satisfied on the chord lines.

Suppose in the $x^*,y^*$ plane there is a staggered cascade of infinitely many thin airfoils. The shape of the airfoils is given by a camber distribution $y_c^*(x^*)$ and a thickness distribution $y_t^*(x^*)$. The contour curves of an airfoil are given by $y_s^*(x^*) = y_c^*(x^*) + y_t^*(x^*)$, $|x^*| < c$, $y_p^*(x^*) = y_c^*(x^*) - y_t^*(x^*)$, $|x^*| < c$ where $y_s^*$ and $y_p^*$ describe the suction and the pressure surface, respectively. $2c$ is the chord length of the airfoils.

The geometric characteristics of the cascade are given by the stagger angle $\lambda$ and the spacing parameter $T = a/c$, where $a$ is the distance between two adjacent airfoils.

We assume that the incoming fluid has a complex velocity $We^{-i\alpha}$ as $x^* \to -\infty$. Let $V_0^*(z^*) = u_0^*(x^*,y^*) - iv_0^*(x^*,y^*)$ be the corresponding basic steady-state complex velocity field, where $z^* = x^* + iy^*$, $[i = (-1)^{1/2}]$.

In addition to the steady-state flowfield there is a small unsteady flowfield caused by oscillatory normal velocities prescribed on the airfoils. The unsteadiness is assumed to be sinusoidal in time and for the sake of convenience will be

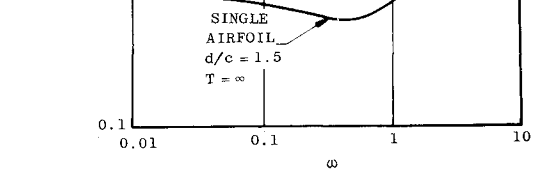


**Fig. 5 Amplitude of normalized lift coefficient; airfoils oscillating about their rear quarter points.**

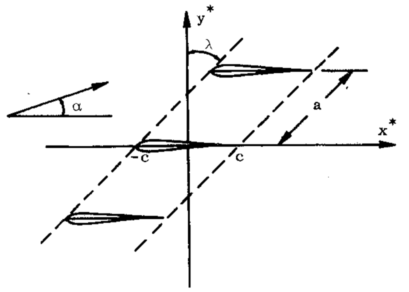


**Fig. 6 Flow through a staggered cascade with an angle of attack.**

treated as real or imaginary part of a complex exponential function.

Let $A_m^*, m = 0, \pm 1, \pm 2, \ldots$ be the $m$th airfoil in the $z^*$ plane, beginning with $A_0^*$ on the $x$ axis and running in positive and negative $y^*$ direction, respectively. We assume on the airfoil $A_m^*$ small normal velocities of the form

$$v_s^*(x^*)e^{j(2\pi\nu t^* - m\mu)}, \quad |x^*| < c, \tag{A1}$$

$$v_p^*(x^*)e^{j(2\pi\nu t^* - m\mu)}, \quad |x^*| < c, \tag{A2}$$

are given, where $v_s^*$ and $v_p^*$ correspond to the suction and the pressure surfaces, respectively. $j = (-1)^{1/2}$ is the second imaginary unit which is introduced in (A1) and (A2) just for convenience and the actual velocities are the real or imaginary parts with respect to $j$. $j$ must be strictly distinguished from $i$, the first imaginary unit introduced to define the complex velocity field and the complex variable $z^*$. For example $ij \neq -1$. $\nu$ is an arbitrary frequency, $t^*$ is the time variable and $\mu$ the phase lag term is given by

$$\mu = 2\pi\kappa/N \tag{A3}$$

where $N \geq 1$ is an integer corresponding to the number of compressor blades and $\kappa$ is an integer equaling the number of cycles of circumferential distortion of the incoming flow, if there are any. $\kappa$ and thus $\mu$ are zero in the case of a purely oscillatory flow. Because of this special form of $\mu$ we can be confined to the airfoils $A_0^*, A_1^*, \ldots A_{N-1}^*$, for, the normal velocities have a period $N$ on the cascade.

Now we wish to find the total complex velocity field

$$V^*(z^*,t^*) = V_0^*(z^*) + V_1^*(z^*)e^{j2\pi\nu t^*} \tag{A4}$$

where $V_0^*(z^*)$ is the solution corresponding to the basic steady state flow mentioned earlier and

$$V_1^*(z^*)e^{j2\pi\nu t^*} = [u_1^*(x^*,y^*) - iv_1^*(x^*,y^*)]e^{j2\pi\nu t^*} \tag{A5}$$

is the superposed unsteady complex velocity. In (A5) $u_1^*$ and $v_1^*$ are complex quantities with respect to $j$. The physically meaningful velocity components are given by the real or imaginary parts of (A5) with respect to $j$ corresponding to the real or imaginary parts of the boundary values (A1) and (A2).

Now we introduce the following nondimensionalized quantities:

$$x = x^*/c, \; y = y^*/c, \; z = z^*/c, \; t = (W/c)t^* \tag{A6}$$

$$\omega = \frac{2\pi\nu c}{W}, \; V_0(z) = \frac{V_0^*(cz)}{W}, \; V_1(z) = \frac{V_1^*(cz)}{W} \tag{A7}$$

$$v(x) = \frac{v^*(x^*)}{W}, \; |x| < 1 \tag{A8}$$

We denote by $A_m$ the $m$th airfoil in the $(x,y)$ plane and by $S_m$ the corresponding streamline beginning at the trailing edge of $A_m$. As an approximation for the unsteady case we neglect the influence of camber and thickness on $S_m$. Therefore, we assume that $S_m$ are the streamlines determined by the basic steady-state velocity flowfield $V_{0\alpha}(z)$. If $S_0$ is given by

$$\zeta_0(s) = \xi(s) + i\eta(s), s \geq 0 \quad \text{(A9)}$$

where $s$ is the arc length of $S_0$ beginning at $z = 1$ and $\xi,\eta$ are the coordinates on the streamline $S_0$. Then $S_m$ is given by

$$\zeta_m(s) = \zeta_0(s) + imTe^{-i\lambda}, m = 0,\pm 1,\ldots \quad \text{(A10)}$$

Following Meister's paper[7] and making some changes which are necessary for our case (in Meister's paper $\alpha = 0$ and $S_m$ are straight lines parallel to the $x$ axis) we can represent $V_1(z)$ as follows:

$$V_1(z) = V(z) + \Gamma\Omega(z) \quad \text{(A11)}$$

where $V(z)$ is a holomorphic function for $z \neq z_m$ which has to be determined, $\Gamma$ is an arbitrary constant and

$$\Omega(z) = \sum_{m=0}^{N-1} \int_0^\infty K(z - \zeta_m)\, e^{-j\mu m} f(s) ds \quad \text{(A12)}$$

with

$$K(z) = \frac{ie^{i\lambda}}{2NT}\left[\coth\left(\frac{\pi e^{i\lambda}}{NT} z\right) + 1\right] \quad \text{(A13)}$$

$$f(s) = -j\omega \exp\left\{-j\omega \int_0^s [d\sigma/W_\alpha(\sigma)]\right\}/W_\alpha(s),\ s \geq 0 \quad \text{(A14)}$$

where $W_\alpha(s)$ is the steady state nondimensionalized speed on the streamlines $S_m$ given by

$$W_\alpha(s) = u_{0\alpha}[\xi(s),\eta(s)]\xi'(s) + v_{0\alpha}[\xi(s),\eta(s)]\eta'(s), s \geq 0 \quad \text{(A15)}$$

where $u_{0\alpha}$ and $v_{0\alpha}$ are the basic steady-state velocity components in $x$ and $y$ directions on the streamline $S_m$. We write

$$V(z) = u(x,y) - iv(x,y), \quad \text{(A16)}$$

$$\Omega(z) = g(x,y) - ih(x,y), \quad \text{(A17)}$$

Then for the determination of $V(z)$ and $\Gamma$ we have the following mathematical problem:

Find a complex function $V(z)$ and a constant $\Gamma$ such that the following conditions are satisfied:

1) $V(z)$ is holomorphic for all $z \neq z_m$ $m = 0, \pm 1, \ldots;$

2) $V(z) = V(z + iNTe^{-i\lambda})$ for all $z$

3) $V(z) = o(1)$ for $x \to -\infty$, $V(z) = 0(1)$ for $x \to +\infty$ (A18)

4) $V(z) = 0[(1 + z)^{-\eta}], 0 \leq \eta < 1$, for $z \to -1$ (A19)

$V(z) + \Gamma\Omega(z) = 0(1)$ for $z \to 1$ (A20)

5) $v(x + kT \sin\lambda, kT \cos\lambda + 0) = v_s(x)e^{-jk\mu} - \Gamma h(x + kT \sin\lambda, kT \cos\lambda)$ (A21)

$v(x + kT \sin\lambda, kT \cos\lambda - 0) = v_p(x)e^{-jk\mu} - \Gamma h(x + kT \sin\lambda, kT \cos\lambda),$ (A22)

for $k = 0,1,\ldots,N - 1;\ |x| < 1$

6) Kelvin's circulation theorem must hold.

The preceding formulation could also be used for finding the solution for basic steady-state velocity field $V_0(z)$.

Meister has shown[7] that the unsteady problem has a uniquely determined solution if $\alpha = 0$ and $v_s(x) = v_p(x)$. But his approach can also be extended to the more general case of the present formulation.

**Reduction to an Integral Equation**

The basic ideas for the solution of the problem are given in Söhngen's paper.[12] We place vortices and sources and sinks on the airfoils $A_m$ and the problem will be reduced to the determination of the densities of these vortices, sources and sinks.

The boundary conditions suggest that we take the densities as

$$[\gamma(x) - i\delta(x)]e^{-jm\mu},\ |x| < 1 \quad \text{(A23)}$$

for $m = 0,1,\ldots,N - 1$ where $\gamma$ corresponds to the vortices and $\delta$ to the sources and sinks.

A straight forward extension of Meister's approach yields

$$V(z) = \sum_{m=0}^{N-1} \int_{-1}^{1} K(z - \xi - imTe^{-i\lambda})\, e^{-jm\mu} \times [\gamma(\xi) - i\delta(\xi)]d\xi \quad \text{(A24)}$$

where the arbitrary constant $\Gamma$ is then determined by

$$\Gamma = \int_{-1}^{1} \gamma(x)dx \quad \text{(A25)}$$

(A24) and (A25) already satisfy conditions (1), (2), (3), and (6) of the problem.

Now, we have to determine $\gamma(x)$ and $\delta(x)$. It can easily be seen that (A24) satisfies the conditions (A21) and (A22) for $k = 1,2,\ldots,N - 1$, if it satisfies these conditions for $k = 0$. This follows from the periodicity properties of (A13) and the special form of (A24). Therefore, we have to satisfy (A21) and (A22) only for $k = 0$.

In order to satisfy the conditions (A21) and (A22) we need the real and imaginary parts of $K(z)$ with respect to $i$. Writing

$$K(z) = R(x,y;\lambda,NT) + iI(x,y;\lambda,NT) \quad \text{(A26)}$$

we have

$$R(x,y;\lambda,NT) = [\{\cos\lambda \sin[(2\pi/NT)(x \sin\lambda + y \cos\lambda)] - \sin\lambda \sinh[(2\pi/NT)(x \cos\lambda - y \sin\lambda)]\}/ 2NT\{\cosh[(2\pi/NT)(x \cos\lambda - y \sin\lambda)] - \cos[(2\pi/NT)(x \sin\lambda + y \cos\lambda)]\}] - \sin\lambda/2NT \quad \text{(A27)}$$

$$I(x,y;\lambda,NT) = [\{\cos\lambda \sinh[(2\pi/NT)(x \cos\lambda - y \sin\lambda)] + \sin\lambda \sin[(2\pi/NT)(x \sin\lambda + y \cos\lambda)]\}/ 2NT\{\cosh[(2\pi/NT)(x \cos\lambda - y \sin\lambda)] - \cos[(2\pi/NT)(x \sin\lambda + y \cos\lambda)]\}] + \cos\lambda/2NT \quad \text{(A28)}$$

It can easily be shown that

$$I(x,o;\lambda,NT) = 1/2\pi x + S(x;\lambda,NT) \quad \text{(A29)}$$

where $S(x;\lambda,NT)$ is analytic around $x = 0$, and that $R(x,o;\lambda,NT)$ has no singularity for $x = 0$.

With (A16), (A24), (A26), (A29) and the application of Plemelj's formulas (see Ref. 11) we obtain the following equations:

$$\gamma(x) = u(x, +0) - u(x, -0) \quad \text{(A30)}$$

$$\delta(x) = v_s(x) - v_p(x) \quad \text{(A31)}$$

where $\gamma(x)$ has to satisfy the following singular integral equation:

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{\gamma(\xi)}{\xi - x} d\xi - N(x) = \frac{1}{2}[v_s(x) + v_p(x)] + \Gamma \sum_{m=0}^{N-1} \int_0^\infty I[x - \xi(s) - mT \sin\lambda, -\eta(s) - mT \cos\lambda;\lambda,NT]e^{-jm\mu}f(s)ds \quad \text{(A32)}$$

with

$$N(x) = \int_{-1}^{1} S(x - \xi;\lambda,NT)\,\gamma(\xi)d\xi + \sum_{m=1}^{N-1}\int_{-1}^{1} I(x - \xi - mT\sin\lambda, -mT\cos\lambda;\lambda,NT) \times e^{-jm\mu}\,\gamma(\xi)\,d\xi - \sum_{m=0}^{N-1}\int_{-1}^{1} R(x - \xi - mT\sin\lambda, - mT\cos\lambda;\lambda,NT)\,e^{-jm\mu}\,\delta(\xi)d\xi \quad \text{(A33)}$$

Equations (A25) and (A32) have to be solved simultaneously to obtain $\gamma(x)$ and $\Gamma$, which will yield the complete solution of the problem. Appendix B deals with the details of finding an approximate solution of these integral equations.

It may be pointed out that in the limit of $T \to \infty$, (A32) yields the corresponding equation for an isolated airfoil.

## Appendix B: Derivation of the Approximate Solution for $\gamma(x)$

The equations for $\gamma(x)$ in the special case of $\mu = 0$, $(N = 1)$; $\alpha = 0$; no thickness, no camber, and sinusoidal traveling wave type of freestream disturbances, are

$$\frac{1}{2\pi}⨍_{-1}^{1}\frac{\gamma(\xi)}{\xi - x}\,d\xi - \int_{-1}^{1} L_1(x - \xi)\gamma(\xi)d\xi + i\omega\Gamma\int_{1}^{\infty} I(x - \xi)\,e^{-j\omega(\xi-1)}d\xi = e^{-j\omega x} \quad \text{(B1)}$$

$$\int_{-1}^{1}\gamma(x)dx = \Gamma \quad \text{(B2)}$$

where

$$L_1(x) = I(x) - \frac{1}{2\pi x}, \quad \text{(B3)}$$

$$I(x) = Im\left\{\frac{ie^{i\lambda}}{2T}\left[\coth\left(\frac{\pi e^{i\lambda}}{T}x\right) + 1\right]\right\} \quad \text{(B4)}$$

We rewrite (B1) as

$$\frac{1}{2\pi}⨍_{-1}^{1}\frac{\gamma(\xi)}{\xi - x}\,d\xi = \frac{j\omega\Gamma}{2\pi}\int_{1}^{A}\frac{e^{-j\omega(\xi-1)}}{\xi - x}\,d\xi - j\omega\Gamma \times \int_{A}^{\infty} I(x - \xi)e^{-j\omega(\xi-1)}d\xi + e^{-j\omega x} + \int_{-1}^{1} \times L_1(x - \xi)\gamma(\xi)d\xi - j\omega\Gamma\int_{1}^{A} L_1(x - \xi)e^{-j\omega(\xi-1)}d\xi \quad \text{(B5)}$$

where $A > 1$ has been introduced to isolate the singularity of $I(x)$ for $x = 0$. For practical purposes $A$ will be chosen in the neighborhood of 4.

In order to get an approximate solution for (B5) we assume for $L_1(x)$ the following polynomial approximations

$$L_1(x) \approx \sum_{k=0}^{n} l_k x^k,\ |x| \leq 2 \quad \text{(B6)}$$

$$L_1(x) \approx \sum_{k=0}^{n} p_k x^k,\ -1 - A \leq x \leq 0 \quad \text{(B7)}$$

The polynomial approximations can be performed in various ways to any desired degree of accuracy by varying $n$. The coefficients $l_k$ are dependent on $\lambda$ and $T$ and $p_k$ on $\lambda,T$ and $A$.

It is well known[10] that the solution $f(x)$ of the following singular integral equation

$$\frac{1}{2\pi}⨍_{-1}^{1}\frac{f(\xi)}{\xi - x}\,d\xi = F(x),\ |x| < 1 \quad \text{(B8)}$$

which is finite at $x = 1$, is given by

$$f(x) = -\frac{2}{\pi}\left[\frac{1 - x}{1 + x}\right]^{1/2}⨍_{-1}^{1}\left[\frac{1 + y}{1 - y}\right]^{1/2}\frac{F(y)}{y - x}\,dy \quad \text{(B9)}$$

Using (B6), (B7) and applying (B9) to Eq. (B5), we obtain

$$\gamma(x) = \left(\frac{1 - x}{1 + x}\right)^{1/2}\left\{\Gamma[c(x) + g(x)] + D(x) + \sum_{k=0}^{n} r_k x^k\right\} \quad \text{(B10)}$$

where

$$c(x) = \frac{j\omega}{\pi}\int_{1}^{A}\left(\frac{\xi + 1}{\xi - 1}\right)^{1/2}\frac{e^{-j\omega(\xi-1)}}{x - \xi}\,d\xi \quad \text{(B11)}$$

$$g(x) = \frac{j2\omega}{\pi}⨍_{-1}^{1}\left(\frac{1 + y}{1 - y}\right)^{1/2}\frac{1}{y - x}\int_{A}^{\infty} I(y - \xi) \times e^{-j\omega(\xi-1)}\,d\xi dy \quad \text{(B12)}$$

$$D(x) = -\frac{2}{\pi}⨍_{-1}^{1}\left(\frac{1 + y}{1 - y}\right)^{1/2}\frac{e^{-j\omega y}}{y - x}\,dy \quad \text{(B13)}$$

$c(x)$, $g(x)$, and $D(x)$ correspond to the first three terms on the right side in (B5), the polynomial $\Sigma r_k x^k$ corresponds to the last two terms of (B5). In (B10), $\Gamma$ and $r_0,r_1,\ldots,r_n$ are still unknown. Therefore, we substitute (B10) together with (B6) and (B7) into (B5) and (B2), and comparing the coefficients of the different powers of $x$ on both sides, we obtain the following system of linear algebraic equations:

$$M\mathbf{Y} = \mathbf{B} \quad \text{(B14)}$$

where

$$\mathbf{Y}^T = (r_0,r_1,\ldots,r_n,\Gamma) = (y_1,y_2,\ldots,y_{n+1},y_{n+2}) \quad \text{(B15)}$$

$$B^T = (d_1,d_2,\ \ldots,\ d_{n+2}) \quad \text{(B16)}$$

$$M = (a_{m,k});\ m,k = 1,\ldots,n + 2, \quad \text{(B17)}$$

$$a_{m,k} = \frac{1}{2\pi}h_{k-m}\,\Theta_{k-m-1} + \sum_{\alpha=m-1}^{n}(-1)^{\alpha-m}\binom{\alpha}{m-1}l_\alpha\Theta_{\alpha+k-m} \quad \text{(B18)}$$

$$m,k = 1,2,\ldots,n + 1,$$

$$a_{m,n+2} = a_{m-1} + b_{m-1} + \sum_{\alpha=m-1}^{n}(-1)^{\alpha-m}\binom{\alpha}{m-1}l_\alpha I_{\alpha-m+1} \quad \text{(B19)}$$

$$m = 1,2,\ldots,n + 1$$

$$a_{n+2,k} = \Theta_{k-1},\ k = 1,2,\ldots,n + 1 \quad \text{(B20)}$$

$$a_{n+2,n+2} = I_0 - 1 \quad \text{(B21)}$$

$$d_m = 2\sum_{k=m}^{n+1}\sum_{r=1}^{k-m+1}(-1)^{k-m}\,l_{k-1}\binom{k-1}{m-1}E_r\Theta_{k-m-r} \quad \text{(B22)}$$

$$m = 1,2,\ldots n + 1$$

$$d_{n+2} = 2\pi E_1 \quad \text{(B23)}$$

The symbols used in the preceding matrix definition are defined as follows:

$$h_k = \begin{cases}1,\ k \geq 0\\ 0,\ k < 0\end{cases} \quad \text{(B24)}$$

$$\Theta_\alpha = \frac{\alpha - 1}{\alpha}\,\Theta_{\alpha-2}\ \text{for even}\ \alpha \geq 2;\ \Theta_0 = \pi \quad \text{(B25)}$$

$$\Theta_\alpha = \frac{\alpha}{\alpha + 1}\,\Theta_{\alpha-2}\ \text{for odd}\ \alpha \geq 1;\ \Theta_{-1} = -\pi \quad \text{(B26)}$$

$$I_\alpha = \int_{-1}^{1} \left(\frac{1-x}{1+x}\right)^{1/2} x^\alpha c(x)\,dx,\ \alpha \geq 0 \qquad \text{(B27)}$$

$$a_m = \sum_{k=m}^{n} \sum_{r=0}^{k-m} (-1)^{k-m} r! \binom{k}{m}\binom{k-m}{r} \times \frac{(1 - A^{k-m-r} e^{-j\omega(A-1)})}{(j\omega)^r} p_k \qquad \text{(B28)}$$

$$m = 0,1,\ldots,n$$

$$b_m = -\frac{j2\omega}{T}(R.P.)_i \sum_{k=0}^{\infty} \sum_{r=m}^{n} \sum_{\alpha=0}^{r-m} (-1)^{r-m} \binom{r}{m} l_r \Theta_{r-m-\alpha-1} \times \frac{e^{i\lambda + j\omega(1-A)}}{(2\pi/T)\, e^{i\lambda}\,(k+1) + j\omega} \beta_{\alpha,k} \qquad \text{(B29)}$$

$$m = 0,1,\ldots,n$$

[(R.P.)$_i$ − real part with respect to $i$]

$$\beta_{\alpha,k} = \frac{1}{\pi} \int_{-1}^{1} \left(\frac{1+y}{1-y}\right)^{1/2} y^\alpha e^{(2\pi/T)e^{i\lambda}(k+1)(y-A)} dy \qquad \text{(B30)}$$

$$\alpha = 0,1,\ldots,n;\ k = 0,1,\ldots$$

$$E_r = \frac{1}{\pi} \int_{-1}^{1} \left(\frac{1+y}{1-y}\right)^{1/2} y^{r-1} e^{-j\omega y} dy \qquad \text{(B31)}$$

$$r = 1,2,\ldots n+1$$

The lift coefficient $C_L(t;\omega)$ defined by Eq. (28) can be calculated as

$$C_L(t;\omega) = \frac{\bar{w}}{W}\left[ j\omega \sum_{k=0}^{n} \Theta_{k+1} r_k(\omega) - j2\pi\omega(E_2 - E_1) - \Gamma(\omega)(1 + j\omega - j\omega I_1)\right] e^{j\omega t} \qquad \text{(B32)}$$

In the last equation, contribution of $g(x)$ which is rather small for $A \simeq 4$ has been omitted. In order to obtain the solution for the steady-state case ($\omega = 0$) the linear system of algebraic equations for $r_0,\ldots,r_n$ is decoupled from the equation for $\Gamma$ and is given by

$$E\mathbf{Z} = \mathbf{B}(0) \qquad \text{(B33)}$$

where

$$\mathbf{Z}^T = (r_0, r_1, \ldots, r_n) \qquad \text{(B34)}$$

$$\mathbf{B}(0)^T = (1,0,\ldots,0) \qquad \text{(B35)}$$

$$E = (e_{m,k}) \qquad \text{(B36)}$$

with

$$e_{m,k} = a_{m,k} \text{ for } m,k = 1,\ldots,n+1 \qquad \text{(B37)}$$

Using the solution of (B33) we can write $\Gamma(0)$ as

$$\Gamma(0) = \sum_{k=0}^{n} \Theta_k r_k(0) \qquad \text{(B38)}$$

Now, the function $H(\omega)$, defined in Eq. (29) is

$$H(\omega) = \frac{1}{\Gamma(0)}\left[\Gamma(\omega)(1 + j\omega - j\omega I_1) + j2\pi\omega(E_2 - E_1) - j\omega \sum_{k=0}^{n} \Theta_{k+1} r_k(\omega)\right] \qquad \text{(B39)}$$

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